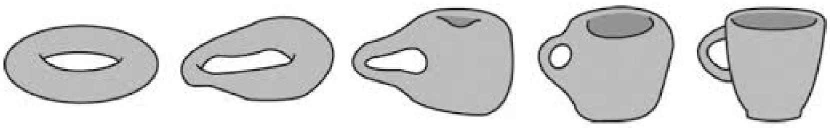
The Figuration of Space

Andrea Ricolfi



Les mathématiciens n'étudient pas des objets,
mais des relations entre les objets;
il leur est donc indifférent de remplacer ces objets
par d'autres, pourvu que les relations ne changent pas.
La matière ne leur importe pas, la forme seule les intéresse.

— H. Poincaré, *La Science et l'Hypothèse (1901)*

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Contents

1 Introduction

The goal of this note is to introduce and describe the problem, arising in Mathematics and Physics, of the figuration of space. The word "space" can have many different meanings in Mathematics. Nevertheless, we will only cope with its *geometric* interpretation. In other words, we will focus primarily on the question:

$\mathscr{Q}$. What is a geometric *space*, and how can we get to know it?

The present note is not intended to give a satisfactory answer to the above question: after all, $\mathscr{Q}$ is nothing but a rephrasing of the (obviously ambitious) question: "What is Geometry all about?". What we would like to do is just to give a flavor of how mathematical ideas have changed in nature during the past century, leading to a new amazing concept of a *space*.

1.1 The main argument

The key idea of our argument is as follows. Suppose we are in a Supermarket, and we suddenly realize that the goal of our life is to understand boxes of oranges. We have one such box right in front of us: let us call it X, say. We wish to understand X as deeply (read: *geometrically*) as possible; we stare at X intensely. We cannot help noticing that a box of oranges consists of a certain collection of... oranges.

But here is the main point: if we just focus on the oranges, as we are tempted to do, we miss the actual geometry: how is X stacked with respect to other boxes? is it in balance? is it straight or slopping? is it on the boundary of a shelf? how are the oranges stacked with respect to one another? Unfortunately, we do lose all this information when we just look at the oranges. Thus, here is the first lesson Geometry teaches us: if X is now a space, and the oranges are the points in X,

$\mathscr{L}_1$. We cannot restrict ourselves to describing the points of X.

Furthermore, understanding the particular X "directly" is not the main goal one should have in mind. Indeed,

- X might be difficult to study; and
- the "essence" of X might be shared by other spaces, close to X in a suitable sense; by sake of cheapness, a geometer cannot help *identifying* all such spaces.

The ideal strategy is to try and relate all X's instead of staring at a particular one: we are not interested in X in itself, but rather in the properties that make X *essentially* what it is. Thus, the second lesson we learn from Geometry is:

$\mathscr{L}_2$. Looking at the single object X is not the correct stategy.

These two lessons will come up again later, as explaining them in detail is one of the goals of this note.

TWO WORDS ON THE VERB *to identify*. When we say we identify two objects under a certain relation, we mean something quite deep. But, hopefully, the following silly comparison may clarify the point: let A be a group of people in a hall; we can define the relation "having the same hair color" on the set A. Denote by $\sim$ this relation, so that for two elements (human beings) $x, y \in A$, we will have $x \sim y$ if and only if x and y have the same hair color. We can of course partition A into smaller groups according to hair color; for instance, for any blonde y, say, we can define the *equivalence class* of y to be

$$[y] = \{ x \in A \mid x \sim y \} = \{ x \in A \mid x \text{ is blonde} \} \subset A.$$

The set of equivalence classes is called the *quotient* of A under $\sim$, and is denoted by

$$A/_{\sim} = \{ [y] \mid y \in A \} .$$

One has to think of it as "A, up to $\sim$". The main point is that, in any equivalence class, any two people give rise to the same class (hence are *identified* in $A/_{\sim}$): well, maybe in A we have Bob $\neq$ Jessica, but all that matters to us was hair color: hence [Bob] = [Jessica], as soon as they have the same hair color. In other words, $x \sim y$ if and only if $[x] = [y]$ as elements in $A/_{\sim}$.

To identify similar boxes, we need to look at *all* of them and to find out the similarities. For instance, we may happen to notice that some boxes contain the same number of oranges: this is one common feature that we certainly cannot ignore. There might be important, more refined (geometric!) properties that our particular X shares with different boxes. And we want to capture them all!

But a question naturally arises from the above: if on the one hand it is easy to check hair color, how can we check that two apparently unrelated geometric objects are in fact quite similar to one another? How can we compare them?

We need a way to relate them to one another, and we have to make sure that, while we do this, the structure is preserved (i.e. that we do not end up by comparing X with a confection of butter). This will be accomplished by the fundamental notion of a *morphism*,[1] which is the main character of this whole work: a morphism is a transformation between... boxes of oranges (in this case), with the property that the very essence of a box (what makes it a box of oranges, instead of a confection of frozen cod) stays preserved: all that matters is the preservation of the structure, by moving from a box (or space) to another.

[1] From ancient greek $\mu o \rho \varphi \eta, \eta \varsigma$: *shape*.

1.2 Guided by the Structure

Let us forget about oranges, which were just a comparison. Suppose we have a space, whatever it is: call it X. It is the kind of object we are interested in. It does have *points*, but looking at points only would mean studying a *set*, which is the most *static* object one can think of (as it will be clearer in the next section). Instead, what makes a "geometry" is exactly the contrary:

$$\text{Geometry} \quad \longleftrightarrow \quad \text{motion, relations, interactions!}$$

The modern viewpoint is that in order to understand a space, it is useless to look at its (motionless!) points only; it is rather meaningful to look at how this space interacts with, or deforms to similar (= with the same structure) spaces. In other words, we will develop the following guiding principle:

$\mathscr{P}$. We understand a certain class of mathematical objects once we understand the *interrelations* between such objects.

The principle $\mathscr{P}$ is the mathematical incarnation of the Philosophical tendency called Structuralism. Interrelations between objects, as we shall see, are governed by *morphisms*, and morphisms belong to the fascinating world of *Category Theory*. We will use categorical language throughout, and this explains the presence of the next section.

Motivations from Physics. We will apply $\mathscr{P}$ to Geometry. But before that, it is convenient to illustrate its power with a bunch of physical examples. First, suppose we have a system of particles, say of gas particles. What do we know about this system when it is at rest? *Nothing*. If we want to capture any relevant information about it, e.g. its mass or some other invariant of it, we have to let the particles interact: they have to be free of running around and possibly collide. When we write the equations of motion, say, we can extract some relevant information. But in order to write them down, we do need motion!

As a second example, consider the recent development of particle accelerators (note the word *accelerator*!): this may shed light on the importance of looking at how the interesting objects (particles) move and interact, in order to get closer to their comprehension.

As a last example, consider String Theory: the fundamental particles are *strings*, which are one-dimensional objects, propagating in a "stringy" space of dimension 6. Motion again! and, most important, the objects of interest do not even have the dimension of a point (zero): they are curves!

Rules of the game. We will see in the next section how Structuralism materializes in the algebraic theory of *categories*: what is important, when studying objects of a given type, is not the single object in itself; rather, its shape, the STRUCTURE of its very essential skeleton, and how such a structure stays preserved.

This concept of "preserving the structure" is essential, and has its own place in the main datum of a category: the arrows, also called the *morphisms*. The unique rule for moving from a space to another one through one such morphism is hidden in the (greek root of the) word itself:

the unique rule is: to preserve the structure, the *shape*.

1.3 This note in a few lines

To face $\mathscr{Q}$ in a suitable way, we have remarked (Poincaré remarked, actually) that the unavoidable notion attached to a *geometry* is that of *interrelations* between similar (yet possibly different) spaces. In fact, two issues leave us tremendously disappointed, when trying to understand a geometric space X:

1. its points, which make X into just a set, and lead us to forget the geometry;
2. the space X itself. Indeed, the geometry of X is better understood in terms of the geometry of other spaces which are related to X (by some suitable morphism).

The goal of what follows is to glance the modern algebro-geometric machinery which makes us much less disappointed about these issues.

2 The structure of mathematical entities

The introduction of the cipher 0 *or the group concept was general nonsense too, and mathematics was more or less stagnating for thousands of years because nobody was around to take such childish steps...*

— A. Grothendieck, *writing to Ronald Brown.*

SUMMARY.

In this section we provisorily forget about our aim: spaces and their geometry. We just focus on the main tool we will use, namely *Category Theory*.

The whole theory of categories is probably the most evident incarnation of the *Structuralism* tendency in Mathematics, more precisely in that field of Pure Mathematics which goes under the name of Abstract Algebra.

Categories constitute a crucial tool in several fields of Mathematics. Not only they provide a neat and elegant language, and powerful theorems which can be applied in a wide variety of situations, but also they form an interesting, yet quite abstract research field in themselves.

Aside 2.1. Category Theory is known, among mathematicians, as *Abstract Nonsense* (which explains the epigraph of this section), a mean but somehow affectionate nickname. After all, the adjective is quite appropriate: what is more abstract than focusing on the relationships between mathematical entities, instead of those entities themselves? For the reader's amusement, we would like to include here part of the last-second page of the book *Reports of the Midwest Category Seminar IV* (Springer-Verlag, 1970). It is a hilarious "Final Examination" in Category Theory: two pages of the most clever irony, after more than one hundred pages of advanced Mathematics. As a note for the reader, the "true founder of Category Theory" according to this page is... Lewis Carroll, the king of poetical nonsense.

CATEGORICALLY, THE FINAL EXAMINATION
FOR THE
SUMMER INSTITUTE AT BOWDOIN COLLEGE (Maine) 1969

'I thought I saw a garden door that opened with a key,
I looked again and found it was a Double Rule of Three,
And all its mysteries, I said, are plain as day to me.'

(Verse by the true founder of
Category Theory)

Important Instruction: This is a take-home exam:
Do not bring it back!

Answer as many as possible at a time.

1. Are foundations necessary? To put it another way, given a chance, wouldn't Mathematics float?

This section is needed because *morphisms* are needed throughout, and the notion of a morphism lives in the categorical setting. This whole note, as hopefully will be clear at the end, is itself *about* morphisms.

Remember that we wish to somehow avoid coping with mere sets. Amazingly, any set is itself a category. That may sound contradictory, as we just (roughly) proposed a slogan like

Categories good, sets bad!

But this is actually a great example to understand *why* we wish to avoid mere sets: it will be clear soon that a set is the worst category one can think of. Indeed, if the core of a category is the class of its morphisms, then a set, viewed as a category, has no morphisms other than the identities (which are forced to be there by the category axioms)! This translates in an elegant way the staticity of sets.

2.1 (Mathematical) Structuralism

Structuralism is a vast paradigm which crosses many fields. As far as we know, the essence of this tendency is the following: suitably defined interrelations between the objects of interest (depending on the field) are the only possible clue towards understanding those objects.

The author is now trying to briefly present a couple of examples of structuralist thought in a more general context than the field of Mathematics. We hope the experienced reader will forgive the author's sloppiness and incompetence.

It does make sense to date back the origins of Structuralism in the field of (modern) Linguistic, initiated by the Swiss Ferdinand de Saussure and his school, in the early twentieth-century. The foundations of Saussure's thought can be found in [6]. His idea of language (*langage*, in french) is that of a structured system, different from mere speech. It is constituted by mutually defining entities, rather than absolute ones. The basic units of langage are, for Saussure, the *linguistic signs*, which one can think of as *links* between what he calls the *signifier* and the *signified*. The latter is abstract in nature: it is a mental creation depending on what the signifier intended... to signify. What is of interest to us is that the correct "objects of study", in order to understand langage, were exactly these linguistic signs: the links. Not the signifier or the signified in themselves, but rather what joins them to one another. However, langage for Saussure was something more sophisticated than a mere list of names linked somehow to things. This is *not* what is meant by a linguistic sign. The following diagram may illustrate his viewpoint:

$$\text{concept} \xleftrightarrow{\text{linguistic sign}} \text{acoustic image.}$$

The name of a concept naturally produces ($\rightarrow$) an acoustic image (a mental representation of that concept: the signified), and conversely an acoustic image is what allows a speaker ($\leftarrow$) to pronounce the name attached to that image; that name is the signifier. Why is all this important to us? Our main objects of study will be links as well, in our sense: we will call such links arrows, or morphisms.

Two (human) milestones in the development of Structuralism are for sure the American Noam Chomsky and the Belgian Claude Lévi-Strauss.

In his book *Syntactic Structures* (cf. [1]), Chomsky insisted, among other things, on grammar's independence of meaning. Syntax is the *structure* he focuses on: interrelations between words matter, to the extent that they go beyond their meaning. Whence the famous nonsensical, yet grammatically correct, sentence in his book:

"Colorless green ideas sleep furiously".

It is by no means a coincidence that the above sentence is (an example of) what is called a *category mistake*.

In his book *The Elementary Structures of Kinship* (cf. [4]), Lévi-Strauss explores the structure of families: not only the relationships between members of the same family, but also interrelations between different families. His monography originated from an expedition in Brasil, with the aim of studying Nambikwara indians. The founding principle of any society, observed Lévi-Strauss, seemed to be incest ban. Why? The reasons given by anthropologists so far were not fully satisfactory, and this motivated his work. Lévi-Strauss observed that it is in regard of respecting this principle that different communities (often more than just two at a time) agreed on organizing weddings between women and men from different groups. This procedure clearly breaks blood relationships, originating ”alliance” relationships instead, which better could guarantee the observance of the incest ban principle. Once more: the interrelations are the actual character, not the single members of a community, or the single community.

Recap. If we wish to understand social phenomena, like the language of a population, or the creation of a society, we need to look at the very structure of the problem. And this structure consists of interrelations between basic entities: words, people, groups of people. Furthermore:

(i) What sort of interrelations we are talking about depends on the context: syntax, or the act exchanging wemen and organizing weddings.

(ii) What these relationships are supposed to preserve also depends on the context: correct grammar, or the incest ban.

In Mathematics. The appearance of Structuralism in Mathematics can be described, in very simple terms, as follows. Intuitionism claimed that a mathematical object deserving this name is something that possesses a certain explicit ”construction”: one cannot speak about something without being able to produce a tangible example. There are mathematical proofs which are constructive, and mathematical proofs which are not. Both types are now officially accepted to be able to *prove existence* of some mathematical entity. It was not so under Intuitionism. Of course, proofs are proofs, but the *abstract* existence of an object used to be highly disregarded. However, the light of Structuralism finally started brighting the mathematical sky: it was no longer possible to hide the presence of abstract structures, permeating several major fields of Mathematics at the same time. The ubiquity and recurrence of certain structures made it natural to look for a global picture of Mathematics, so to encompass all relevant objects at the same time: mathematicians started looking at the class of *all* sets, of *all* groups, of *all* vector spaces... In other words, mathematical entities were divided into ”classes” (not sets: Russell lies in wait for us!), according to the very structure of the underlying objects.[2] The study of how the

[2] The term *class* is a technical one. The reader can think of a class as the

same structure can appear in different objects *is* the notion of a morphism between objects in the same class. Category Theory was born.

2.2 Categories in a nutshell

A friendly introduction to Category Theory, which is not aimed at mathematicians only, is the book [7], which to our knowledge is not available in a paper cover form (yet). Categories were introduced in [2] in the 1940's. Exactly in the same period, the notation

$$f : X \to Y$$

was introduced to denote a function f from a set X to a set Y. To quote the reference where we learnt this fact: in [5], the author says:

> The fundamental idea of representing a function by an arrow first appeared in topology about 1940, probably in papers or lectures by W. Hurewicz on relative homotopy groups. (cf. [3].) His initiative immediately attracted the attention of R. H. Fox and N. E. Steenrod, whose [8] paper used arrows and (implicitly) functors [...]. The arrow $f : X \to Y$ rapidly displaced the occasional notation $f(X) \subset Y$ for a function. It expressed well a central interest of topology. Thus a notation (the arrow) led to a concept (category).

This seems just a notational issue, but it really is a milestone ("fundamental idea") in the change of perspective that animated the subsequent decades.

Morally, or romantically, one might also date back the invention of categories to the French mathematician and philosopher of Science Henri Poincaré. The epigraph of this note is taken from his work "Science and Hypothesis" (1901), and will serve us as a "slogan" not only for this section but for the rest of this work. Thus it seems adequate to give the English translation of his words, before starting out with the theory of categories:

Mathematicians do not study objects, but the relations
between objects; to them it is a matter of indifference
if these objects are replaced by others,
provided that the relations do not change.
Matter does not engage their attention,
they are interested in form alone.

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blowing-up of a set: something in some sense too big to still be a set. Classes are important in some foundational aspects of Mathematics, e.g. Gödel-Bernays-Von Neumann (BNG) Set Theory, especially when one wants to avoid Russell's Paradox (as we do). Classes are used to define categories. Just to give a flavor of how classes generalize sets, and bypass Russell: one of the basic axioms in BNG says that a class is a set if and only if it belongs to another class.

Giving the precise definition of a category goes beyond the scope of this note. We cannot avoid, however, mentioning its main features. We collect them in the following (incomplete) list:

- Nearly any class of interesting objects in Mathematics form a category.
- A category $\mathscr{C}$ consists of two main data: a class $\mathrm{Ob}\,\mathscr{C}$ of objects and a class $\mathfrak{mor}_{\mathscr{C}}$ of arrows (or morphisms). Objects will be denoted $X, Y, Z \ldots$ Arrows relating the objects will be denoted $a : X \to Y$, $b : Y \to Z, \ldots$
- For each couple of objects X, Y, there is a (possibly empty) set, called a hom-set, consisting of arrows between the two objects (in a fixed direction):

  $$\hom_{\mathscr{C}}(X, Y) = \{\, \text{arrows } X \to Y \,\} \subset \mathfrak{mor}_{\mathscr{C}}.$$

  Each hom-set is a set, but the collection $\mathfrak{mor}_{\mathscr{C}}$ of all arrows is in general a proper class.
- For every $X \in \mathrm{Ob}\,\mathscr{C}$, one hom-set is never empty: $\hom_{\mathscr{C}}(X, X)$ always contains at least an arrow $1_X : X \to X$, called the identity arrow of X.
- Morphisms can be "composed" (in an associative way) with one another:

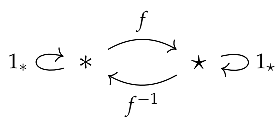
  $$X \xrightarrow{a} Y \xrightarrow{b} Z$$

  has to make sense as a morphism, denoted $b \circ a$, from the object X to the object Z.
- There is a particular kind of arrows which we will be most interested in: *isomorphisms*. We say that an arrow $f : X \to Y$ is an isomorphism when there exists another arrow $g : Y \to X$ such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Necessarily g, called the inverse of f, is an isomorphism too.
- The key idea of a category, which is the reason why we are focusing on them for our purpose, is that *objects do not really matter*: what is important, in understanding the class $\mathrm{Ob}\,\mathscr{C}$, is how objects interact:

  $$\text{arrows } (\mathfrak{mor}_{\mathscr{C}}) \text{ is all that matters!}$$

Examples.

1. A category $\mathscr{C}$ with two distinct but isomorphic objects. It can be represented as follows.



Note how abstract this category can be: no-one told you that $*$ or $\star$ have the structure of a set, for instance. In particular they may have no "elements", and the arrows in such a category are not necessarily *functions*.

2. We will denote by **Ens** the category whose objects are sets, and whose morphisms are functions between sets: just ordinary maps.

3. From *Algebra*: the category of groups, Abelian groups, rings, monoids, fields, vector spaces...

4. From *Geometry*: the category of topological spaces, Lie groups, differentiable manifolds, complex manifolds, schemes, affine schemes...

REMARK 2.1. It is important to notice that two objects $X, Y \in \operatorname{Ob}\mathscr{C}$ in a category $\mathscr{C}$ could be isomorphic in $\mathscr{C}$ but not isomorphic in other categories (where they still appear as objects there, of course). For example, any two sets with 4 elements are isomorphic as sets; but, on the other hand, there are two non-isomorphic groups of order 4, and in the category of groups, no-one will ever be allowed to think of them as "more or less the same group". They cannot be *identified*.

2.3 Sets are rigid

It is well known that a set is characterized by its elements. Recall the lesson $\mathscr{L}_1$ which we learned from Geometry: never restrict yourself to look uniquely at points in a space X (i.e. elements in the underlying set of points of X).

The goal of this subsection is to illustrate the staticity of sets using the language of categories.

It can be proved that a *set*, i.e. an object

$$A \in \operatorname{Ob}\mathbf{Ens},$$

is the same thing as a category where the objects are the elements in A, and the arrows are just the identity arrows: one for each object. This is really saying that a set is the most static object we can think of: when viewed as a category, it only has the bare minimum amount of arrows that the axioms of a category require: the identity arrows. A set of n elements

$$A = \{ x_1, x_2, \ldots, x_n \}$$

could hence be represented as follows, as a category:

$$x_1 \bullet \circlearrowleft 1_{x_1} \qquad x_2 \bullet \circlearrowleft 1_{x_2} \qquad \ldots \qquad x_n \bullet \circlearrowleft 1_{x_n}$$

This means that there are no morphisms from an object (element) to another:

$$\hom_A(x_i, x_j) = \begin{cases} \varnothing, & \text{if } i \neq j \\ \{ 1_{x_i} \}, & \text{if } i = j. \end{cases}$$

2.4 Categories defeat Russell

This subsection is about Russell's paradox. We want to recall it to the reader in order to underline the limitations of sets, opposed to the strength of categories. Moreover, we will use it as a pretext to introduce the notion of a *functor*, or morphism of categories.

Russell's Paradox. The set $\mathscr{S} = \{y \mid y \text{ is a set}\}$ is a paradoxical object.

Proof. If such a set $\mathscr{S}$ existed, we could consider the subset

$$\mathscr{S}' = \{y \in \mathscr{S} \mid y \text{ is not a member of itself}\} \subset \mathscr{S}.$$

If $\mathscr{S}'$ is a member itself, then by definition it is not a member of itself, and conversely. □

On the other hand,

The category of categories makes perfect sense.

Why is that? Let us try to define the category $\mathfrak{Cat}$ of all categories. Firstly, in such an immense category, an object is an ordinary category. This is pretty fine, but what is an arrow in this category? What is a morphism of categories?

Digression about *functors*. Although Category Theory itself belongs to the Field of Abstract Algebra, it is so powerful that its techniques carry over many other fields in Mathematics. We also mentioned that nearly anything which interests mathematicians can be made into a category. One can also move from a category to another, using the notion of a *functor*, which can be thought of as a *morphism of categories*. A functor $F : \mathscr{C} \to \mathscr{D}$ takes objects $X \in \text{Ob}\,\mathscr{C}$ to objects $FX \in \text{Ob}\,\mathscr{D}$, arrows $f : X \to Y$ in $\mathscr{C}$ to arrows $Ff : FX \to FY$ in $\mathscr{D}$, and the category axioms (in other words, the category structure) stays preserved under these assignments. It would be grotesque if we tried to explain (in non-technical words) what a functor is, better than it is explained in [7], thus we limit ourselves to quote a few lines from the Introduction of that excellent book:

> These categories can then be connected together by functors. And the sense in which these functors provide powerful communication of ideas is that facts and theorems proven in one category can be transferred through a connecting functor to yield proofs of analogous theorems in another category. A functor is like a conductor of mathematical truth.

To sum up, $\mathfrak{Cat}$ is a category because functors exist. And nothing like Russell's paradox will ever affect the granite structure of categories.

Recall that one of the appearences of Structuralism is in Linguistics (N. Chomsky). This lead us to make the following:

Unimportant but funny comparison. A functor between two categories can be thought of as *translation* between two languages. What can be formulated in the source language can be translated in the target language, and the process of doing so preserves the structure, meaning that a sensible sentence gets sent to a sensible sentence. Of course, some information can be lost in the process, like for instance when one translates poetry: the perfect balance of metric, and the admirable rhymes constructed by Dante will never be recovered in *any* other language. Not even an echo of them. Something of this sort happens for instance when one starts with a *concrete category* $\mathscr{C}$, i.e. one where the objects are sets. Then, by definition there is a functor

$$F : \mathscr{C} \longrightarrow \mathbf{Ens}$$

to the category of sets, called a *forgetful functor*: it is the functor which "forgets" (literally!) the additional structure present in the objects of $\mathscr{C}$. It intentionally destroys all the information which was present in $\mathscr{C}$.

There is, however, an important point which breaks the comparison between functors and translations: no translation can be fully satisfactory, but some functors can. The example of poetry was just an extreme one. Actually, it is likely that one will never find two languages where the correspondences between words (the objects) and sentences (words linked together: the morphisms) make the two languages "isomorphic". It may very well happen that, after translating a word, translating it back in the source language does not produce the word one started with *only*. There is also another issue: languages carry a heavy structure, due to the *meaning* that each word has. And there will always be a word which has more than one meaning in one language, but only one in the other. Likewise, there will always be a sentence, perhaps a way of saying, which is meaningful in a language but (e.g. for historical reasons) has no counterpart in the other language. Instead, unlike for translations, some functors are *isomorphisms of categories*: they make the two categories essentially the same.

3 Space = Points + Geometry

Summary.

This section is really (trying to be) about Geometry. The goal is now to illustrate, in its full power, the appearance of mathematical Structuralism in the geometric setting. More precisely, we wish to use ideas from *Topology* in order to convince the reader that

only arrows matter.

And in particular, for geometric purposes, it is of extreme importance to focus on *isomorphisms*, and to able to *identify* isomorphic objects.

To phrase it again differently: the goal of this section is to explain why every mathematician *should* consider a cup and a donut as the same thing.

3.1 Topology: the Geometry of rubber

The fundamental object in Topology is called a *topological space*. It is a set, whose elements are called points, but the points, as we already know, do not really matter. For instance, it may very well happen that the same set can be endowed with different topological structures, whatever that means. Well, if points do not determine the topology, what does? The answer is: *open sets*. To give a topology on a set X means to give a certain collection of subsets of X, satisfying some very natural axioms. We do not dig into the details of what these axioms say. The point is that the open sets tell us many things about X, since they define the topology! Notice that each open subset U of X is itself a topological space, and comes with a natural inclusion $U \to X$, which is a morphism in the category of topological spaces.

How to think about these open sets? Take this example: I can declare the open sets to be all the singletons $\{x\}$, where x is any point in X. This I can really do: it is called the *discrete topology* on X. One cannot make smaller open sets. But then studying X as a topological space is nothing but studying X as a set, and this is not interesting, as we already motivated.

Let us think as follows: just for explanation purposes, let us figure out X as a sheet of paper, with its structure being hidden in the very details of the paper; one figures the stressed topologist trying to understand X from above... but his eyes are not strong enough, and his glasses do not help. Then he uses the only tool he has available: the open sets! these can be thought of as a collection of magnifying lenses spread out all over the paper. Some are powerful, some are less, and the way they intersect gives local information about the points in the intersection.

Remark 3.1. To have lenses "of null radius" amounts to have no lenses at all. But these lenses are the only tool to understand X. Hence, in the discrete topology, there is nothing to understand!

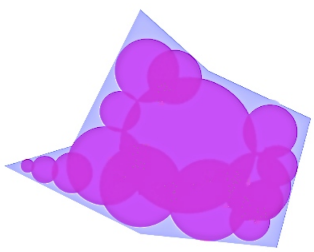


Figure 1: The magnifying lenses, *alias* the open sets.

It goes without saying that topological spaces form a category. The arrows between topological spaces are the *continuous functions*, i.e. those arrows $f : X \to Y$ such that $f^{-1}(U)$ is open in X for every open subset U of Y.

Definition 3.1 (Homeomorphism)**.** An *isomorphism* in this category is a bijective continuous map whose (set-theoretic) inverse is also continuous. Such maps are called *homeomorphisms*. They can also be characterized as bijective continuous maps sending opens to opens.

We will give two examples of homeomorphisms. The idea of a homeomorphism $f : X \to Y$ between two spaces X and Y is that we can *continuously deform* X into Y. This is the reason why Topology is often called

The Geometry of rubber.

Homeomorphic objets tend to share many many properties, so from a geometric point of view it is perfectly legitimate to (mentally) identify such objects. This way, we are reduced to study a smaller class of objects: the *homeomorphism classes* arising from all possible identifications (remind the example of hair color: exactly the same principle). The following is an example of what kind of nontrivial property can be preserved under homeomorphisms.

Example 3.1. This is an example of a famous theorem, one of the favorites by the author. The *Hairy ball theorem* states that there cannot exist a continuous nowhere vanishing vector field on the sphere. This, intuitively, means that if you try to comb a sphere, you can do it of course, but you cannot avoid creating at least a cowlick somewhere. The *continuous vector field* is the configuration of hair on the sphere, and the cowlick is the point on the sphere where the chosen vector field (coiffure) has to vanish. This has also a meteorological interpretation: if now the sphere is the Earth, and the vector field is the wind of a hurricane, then the theorem is saying that there must be a point on Earth where the speed of the wind id 0. This is sometimes called the *eye of the cyclone*. The amazing fact about this theorem, and the reason why we stated it, is that it does not hold for spheres only, but for *any* topological space which is homeomorphic to a sphere. This should shed light on the importance of considering homeomorphism classes.

To conclude, homeomorphisms are to be considered, as they carry important and nontrivial invariants. Knowing this can be a tool to decide whether or not two spaces X, Y are homeomorphic. Indeed, suppose you have calculated, for both X and Y, a certain invariant. If they are the same, then X and Y *might* be homeomorphic. But if they are not, then certainly X and Y *cannot* be homeomorphic. For example, on a torus (later called a "donut"), there exist continuous nowhere vanishing vector fields. *Hence*, by the Hairy Ball Theorem, a donut cannot be homeomorphic to a sphere.

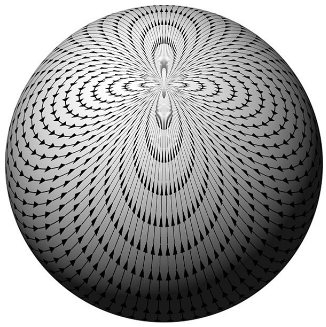


Figure 2: The eye of the cyclone.

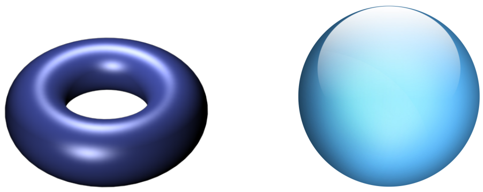


Figure 3: The torus is not homeomorphic to the sphere.

3.2 What is the Earth with the North Pole removed?

In this subsection, we build an example of a morphism between topological spaces. It will be easy to represent it with a picture, and it will be extremely clear that such morphism is in fact an isomorphism in the category of topological spaces. It is known as the *Stereographic Projection*.

Everything happens in the topological space

$$\mathbb{R}^3 = \{ (a,b,c) \mid a,b,c \in \mathbb{R} \},$$

consisting of triples of real numbers. It is the usual Euclidean 3-space. In $\mathbb{R}^3$, we look at the following 2-dimensional subsets: the plane $z = 0$, which can be represented as

$$\mathcal{P} = \{ (a,b,0) \mid a,b \in \mathbb{R} \} \subset \mathbb{R}^3,$$

and the shpere

$$\mathbb{S}^2 = \{ (x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z-1)^2 = 1 \} \subset \mathbb{R}^3$$

of radius 1, centered at the point $(0,0,1)$. On the sphere there is the point $N = (0,0,2)$, which we call the North Pole. As shown in the picture below, we can build a function

$$F : \mathbb{S}^2 \setminus \{ N \} \to \mathcal{P}$$

sending a point $p \in \mathbb{S}^2$ on the sphere to a precise point $F(p) \in \mathcal{P}$ in the plane: the one obtained by intersecting the plane with the

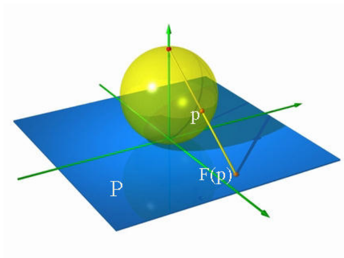
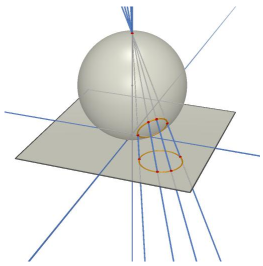


Figure 4: The Stereographic Projection $F : \mathbb{S}^2 \setminus \{ N \} \to \mathcal{P}$.

unique line joininig N and p. For instance, the origin in the plane will be the image of the South Pole (the origin itself, in the picture).

Of course, this map is not defined at N: it is indeterminate, because there are infinitely many lines ”joining N to itself”.[3]

It is also clear that the map F does not respect the lenghts: a small ball centered at a point on the sphere can become a huge open disc in the plane (take such a ball close to the North Pole, to figure what happens).



Continous maps are allowed to strecht and bend (but not cut!) the rubber they are deforming. This is not a problem. All they are required is to do so in a *continuous* manner. The map F, even though it does not respect lengths, is bijective, continuous and sends open sets to open sets, hence it is a homeomorphism. To sum up, and to answer the question in the title of this subsection: from a topological perspective, the Earth with the North Pole removed is the same thing as a plane.

3.3 Donuts and cups

This very short subsection is quite recreational, as a serious example of a homeomorphism has been given in the previous subsection. We will now make sense of the joke of which topologists have been the (legitimate) victims since a long time ago: they are indeed laughed at because they are unable to[4] distinguish between a donut

[3] This indeterminacy can be ”resolved” by passing to the projective closure of the plane, i.e. by adding the so called ”line at infinity”, a one-dimensional object whose points capture all the lines through N, among which we were unable to choose inside $\mathbb{R}^3$.

[4] And they have to be unable!

and a cup. The reason is that there a homeomorphism between these two geometrical spaces, meaning that one is able to deform continuously one into the other, and this deformation is bijective and respects the open sets. One really has to think as these spaces as being made of rubber! Here is a picture of the deformation:

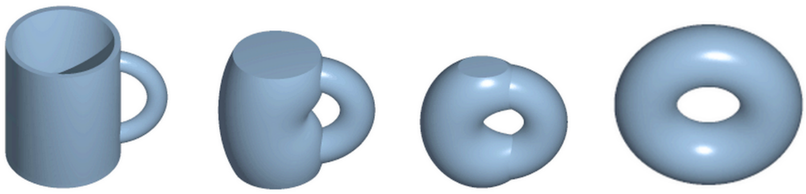


Figure 5: Cup $\cong$ donut.

4 The power of arrows

SUMMARY.

The two parts of this section will try to cope with the lessons $\mathscr{L}_1$ and $\mathscr{L}_2$, respectively, we "learnt" in the Introduction. We will talk (inappropriately, i.e. giving no definition at all) about the modern algebro-geometric notion of a *space*, a structure going under the name of a *scheme*.

In order to fix $\mathscr{L}_1$, we will give the example of the (affine) scheme $X = \operatorname{Spec}\mathbb{Z}$: we will be able to represent any of its *points* by means of a genuine *morphism*. This will enable us to get rid of the "static" intuition we used to have about points.

In order to fix $\mathscr{L}_2$, we will encounter Yoneda's Lemma, a deep theorem in Category Theory. We will adapt it to our situation, where it will tell us that the mere datum of a scheme X is equivalent to specifying *all morphisms* $Y \to X$, where Y is any other scheme. This is exactly what we announced earlier when dealing with this (structuralist) issue: a space is best understood in terms of the morphisms relating it to other spaces.

4.1 The dynamic nature of points

This subsection addresses lesson $\mathscr{L}_1$ and solves (up to the reader's faith in the presented example) the problem of the staticity of points.

We will explain "the dynamic nature of points" by means of an extended example, which generalizes widely.

The goal is to convince the reader that in suitable spaces, called schemes, which are locally modelled on affine schemes (see below),

a point *is* a morphism.

Let us get started now.

Rings and their ideals. Let $\mathbb{Z}$ be the ring of integers: as a set, it is just

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

But a *ring* is more than just a set: the ring structure remembers the two usual operations $(+, \cdot)$ defined on the integers.

Every prime number $p \in \mathbb{Z}$ generates, under multiplication by any other integer, what is called a *prime ideal* in $\mathbb{Z}$. It is denoted

$$(p) = \{\, pn \mid n \in \mathbb{Z} \,\} \subset \mathbb{Z}.$$

In rigorous terms, one can make the following:

Definition 4.1. A prime ideal in a ring R is any additive subgroup $I \subset R$ such that $IR \subset I$ (so far we have defined *ideal*) and whenever a product xy lies in I, at least one between x and y lies in I itself.

Note that $(p) = (-p)$, so the sign is not an issue. However, there is a prime ideal which is not generated by a prime: the (0) ideal, consisting of the unique number 0. The zero ideal (0) is prime because in $\mathbb{Z}$ the product cancellation law holds: if a product of integers is 0, one of the factors was already 0. Hence $(0) \subset \mathbb{Z}$ is prime by Definition 4.1.

Let us give a name to the set of prime ideals in $\mathbb{Z}$. We call it the *spectrum* of the ring of integers:

$$\operatorname{Spec}\mathbb{Z} = \{\, \text{Prime ideals in } \mathbb{Z} \,\} = \{\, I \subset \mathbb{Z} \mid I \text{ is a prime ideal} \,\}.$$

This is not just a set: first of all it is a *topological space*. Recall that we encountered topological spaces earlier: the sphere, the plane, the sphere minus the North Pole, the donut... We will not spend time to talk about the particular topology[5] that we have on $\operatorname{Spec}\mathbb{Z}$, because much more important structure is present. (We will not talk about this structure, either, but the goal of this discussion is to see its effects.) In other words, we are dealing with a more sophisticated (and complicated) object than the donut, the cup, appeared in the previous section. Any scheme carries a natural algebro-geometric structure: it is a space (geometry!) encoding a huge amount of arithmetic (algebra!) data.

The space $\operatorname{Spec}\mathbb{Z}$ is one of the building blocks of the modern notion of space: $\operatorname{Spec}\mathbb{Z}$ is an example of what is called an *affine scheme*.[6] Any *scheme*, locally in its topology, looks like an affine scheme, and every affine scheme is by definition the spectrum of a ring. We now wish to convince the reader that such "points" are not static at all: they are morphisms themselves, as it will be made clear by this simplified example.

[5] This is called the *Zariski topology*.

[6] Scheme Theory was introduced by Alexander Grothendieck in the 1950's. It goes without saying that affine schemes form a category.

To explain the dynamic nature of points, we will use the instance of $\operatorname{Spec}\mathbb{Z}$, which we now rename X.

Background material.

§ **I**. The field $\mathbb{F}_p$ of integers modulo p.

Remind the example of the relation "having the same hair color". Now we do the same with the relation, to be imposed on the integers $\mathbb{Z}$, "having the same remainder in the division by p". In other words, we consider equivalent two integers n and n' when $n - n'$ is a multiple of p. We call such n and n' *equivalent modulo* p. Of course, the possible remainders in the division by p are $0, 1, \ldots, p-1$. This says there are p equivalence classes of integers modulo p, namely

$$[i] = \{\, n \in \mathbb{Z} \mid n - i \text{ is a multiple of } p \,\} \subset \mathbb{Z},$$

for $0 \leq i \leq p-1$. These form a set

$$\mathbb{F}_p = \{\, [0], [1], [2], \ldots, [p-1] \,\},$$

which is not only a set,[7] but it is a (finite) *field*, i.e. a commutative and unitary ring in which every nonzero element is invertible. It is the *unique* field, up to isomorphism, containing exactly p elements, and it can be constructed as a quotient of $\mathbb{Z}$, as indicated above (dividing out $\mathbb{Z}$ by the overquoted relation). Thus, once more, each $[i] \in \mathbb{F}_p$ should not be considered as the integer i, but as "i, up to any multiple of p": $[i]$ is a "representative" of *all* those integers that you can write as $i + tp$, for some $t \in \mathbb{Z}$.

Example 4.1. If $p = 5$, then $n = 12$ and $n' = 27$ are equivalent modulo p, meaning that $n - n' = -15$ is a multiple of p. Both n and n' can thus be denoted by the symbol $[2]$ when viewed inside $\mathbb{F}_5$:

$$12 \bmod 5 = 27 \bmod 5 = [2] \in \mathbb{F}_5.$$

Thus each $\mathbb{F}_p$ is a quotient of $\mathbb{Z}$, i.e. a set of equivalence classes for a relation[8] on $\mathbb{Z}$.

§ **II**. The field $\mathbb{Q}$ of rational numbers.

There is another field we can get starting from $\mathbb{Z}$: the field $\mathbb{Q}$ of rational numbers. Its elements are ordinary fractions, i.e. actual numbers of the form a/b, where $a, b \in \mathbb{Z}$ and $b \neq 0$. The procedure which gives us $\mathbb{Q}$ out of $\mathbb{Z}$ is "inverting" every nonzero integer. This is clear! In more algebraic terms, $\mathbb{Q}$ is the *localization of* $\mathbb{Z}$ *at the ideal* (0).

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[7] Notice how many times we are forced to use the expression "... is not only a set".

[8] One should be more careful and talk about an *equivalence relation*, the only kind of relation which allows to "take quotients".

We are almost there. We need one more definition: for every point $x \in X = \operatorname{Spec} \mathbb{Z}$, define

$$\kappa(x) = \begin{cases} \mathbb{F}_p & \text{if } x = (p), \\ \mathbb{Q} & \text{if } x = (0). \end{cases}$$

Claim. To give a *point*

$$x \in X$$

is the same as to give a morphism of rings

$$\phi_x : \mathbb{Z} \to \kappa(x).$$

Proof. Let $x \in X$ be a point. Let us try to produce a morphism $\mathbb{Z} \to \kappa(x)$. First of all, a point $x \in X$ is a prime ideal $I \subset \mathbb{Z}$, so it is either of the form (p) for a prime $p \in \mathbb{Z}$, or it is the zero ideal $(0) \subset \mathbb{Z}$. Now, we said that each $\mathbb{F}_p = \kappa(p)$ is a quotient of $\mathbb{Z}$: this means there is a canonical (surjective) ring homomorphism

$$\begin{aligned} \pi_p : \mathbb{Z} &\to \mathbb{F}_p = \kappa(p) \\ i &\mapsto [i] = i \bmod p. \end{aligned}$$

On the other hand, $\mathbb{Q} = \kappa(0)$ is a localization of $\mathbb{Z}$, and as such it has a canonical (injective) ring homomorphism

$$\begin{aligned} j : \mathbb{Z} &\to \mathbb{Q} = \kappa(0) \\ n &\mapsto n/1. \end{aligned}$$

Conversely, there is exactly one ring homomorphism from $\mathbb{Z}$ to each of the fields $k(x)$.[9] Thus it has to be the one just defined, for each $x \in X$. Summing up, the correspondence is

$$x \longleftrightarrow \phi_x = \begin{cases} \pi_p & \text{if } x = (p), \\ j & \text{if } x = (0). \end{cases}$$

This finishes the proof. □

More Generally...

As we said at the beginning of this subsection, the instance we described generalizes widely. Indeed, to give a point in an affine scheme $Y = \operatorname{Spec} R$, i.e. to give a prime ideal $\mathfrak{p} \subset R$, *always* boils down to giving a morphism $R \to \kappa(\mathfrak{p})$, where $\kappa(\mathfrak{p})$ is a field canonically attached to the point $\mathfrak{p} \in Y$.

What happens in general is that, given two rings R, S, we have the following bijection between hom-sets:

$$\hom_{\mathbf{Ring}}(R, S) \cong \hom_{\mathbf{Afs}}(\operatorname{Spec} S, \operatorname{Spec} R). \tag{1}$$

[9] This is a general fact: any ring R is a $\mathbb{Z}$-algebra in precisely one way, i.e. $\mathbb{Z}$ is the initial object in the category of rings.

We have denoted by **Ring** the category of rings, and by **AfS** the category of affine schemes.

If we look at our example of $X = \operatorname{Spec}\mathbb{Z}$ once more, we find

$$\hom_{\mathbf{Ring}}(\mathbb{Z}, k(x)) \cong \hom_{\mathbf{AfS}}(\operatorname{Spec}\kappa(x), \operatorname{Spec}\mathbb{Z}).$$

In the left hom-set, there is just one element (as there exists exactly one ring homomorphism from $\mathbb{Z}$ to any other ring). The element

$$\operatorname{Spec}\kappa(x) \to \operatorname{Spec}\mathbb{Z}$$

corresponding to it in the right hom-set *is* the (inclusion morphism of the) point $x \in X$.

Finally, observe the interaction between Algebra and Geometry: the left hom-set in (1) is algebraic (rings are algebraic objects), while the right hom-set is geometric (spectra are geometric spaces).

4.2 Yoneda's Lemma

This subsection addresses lesson $\mathscr{L}_2$ in the Introduction. The goal is now to illustrate Yoneda's Lemma, whose content can be summarized as follows: in a category $\mathscr{C}$, an object X is identified by the functor $h_X : \mathscr{C} \to \mathbf{Ens}$ defined on objects by:

$$T \mapsto h_X(T) = \hom_{\mathscr{C}}(T, X).$$

In other words, any objects is *completely described* by the arrows in $\mathscr{C}$ pointing at it.

Let us fix a category $\mathscr{C}$. Recall that $\hom_{\mathscr{C}}(X, Y)$ is a set for every two objects $X, Y \in \operatorname{Ob}\mathscr{C}$. We now build a new category, which we denote by

$$\underline{\hom}(\mathscr{C}, \mathbf{Ens}),$$

defined as follows: the objects are the functors $\mathscr{C} \to \mathbf{Ens}$ from $\mathscr{C}$ to the category of sets **Ens**. A morphism between two functors (with the same source and target category) is called, in general, a natural transformation. For sake of completeness, we now give the precise definition.

Definition 4.2. Let $F, G : \mathscr{A} \rightrightarrows \mathscr{B}$ be functors. A NATURAL TRANSFORMATION $\eta : F \Rightarrow G$ between F and G is the datum of a subclass of arrows in $\mathscr{B}$,

$$\{\, \eta_A : FA \to GA \,\}_{A \in \operatorname{Ob}\mathscr{A}} \subset \mathfrak{mor}_{\mathscr{B}},$$

which is indexed by $\operatorname{Ob}\mathscr{A}$ and has the following property: for every arrow $f : A \to B$ in $\mathscr{A}$, there is a commutative diagram

$$\begin{array}{ccc} FA & \xrightarrow{\eta_A} & GA \\ \big\downarrow{\scriptstyle Ff} & & \big\downarrow{\scriptstyle Gf} \\ FB & \xrightarrow{\eta_B} & GB. \end{array}$$

Now we have two categories:

$$\mathscr{C}, \text{ and } \underline{\hom}(\mathscr{C}, \mathbf{Ens}).$$

The second one might seem a bit strange, but it does have genuine objects and morphisms, as we just saw. It is often called a *functor category*. Now we define a functor between these two categories: we let

$$F : \mathscr{C} \to \underline{\hom}(\mathscr{C}, \mathbf{Ens})$$

be the functor which takes an object $X \in \operatorname{Ob} \mathscr{C}$ to the functor $h_X \in \operatorname{Ob} \underline{\hom}(\mathscr{C}, \mathbf{Ens})$ in the target category, defined previously. This is a perfectly nice functor. How is it defined on arrows? Well, the image Ff of an arrow $f : X \to Y$ in $\mathscr{C}$ is by definition the following natural transformation (arrow in $\underline{\hom}(\mathscr{C}, \mathbf{Ens})$): the image of f,

$$Ff : h_X \Rightarrow h_Y$$

is defined by the datum

$$\{ (Ff)_T : h_X(T) \to h_Y(T) \}_{T \in \mathscr{C}} \subset \mathfrak{mor}_{\mathbf{Ens}},$$

where the arrow $(Ff)_T$ is nothing but composition by f in $\mathscr{C}$:

$$\begin{aligned} (Ff)_T : \hom_{\mathscr{C}}(T, X) &\to \hom_{\mathscr{C}}(T, Y) \\ \phi &\mapsto f \circ \phi. \end{aligned}$$

We are ready to state:

Lemma 4.1 (Yoneda's Lemma)**.** The functor F is an embedding of categories.

This just means what it looks like: we can regard the source category $\mathscr{C}$ as a subcategory of the functor category $\underline{\hom}(\mathscr{C}, \mathbf{Ens})$. And we do this via the functor F, i.e. by assigning to any object X the functor $FX = h_X$. In particular, Yoneda's Lemma is saying that, for any two objects $X, Y \in \operatorname{Ob} \mathscr{C}$, there is a bijection

$$\hom_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \hom_{\underline{\hom}(\mathscr{C}, \mathbf{Ens})}(h_X, h_Y).$$

Clearly, this map is given by the one we just described, namely $f \mapsto Ff$.

Back to our situation. We are interested in algebro-geometric categories. For instance, if we take as $\mathscr{C}$ the category $\mathfrak{Sch}$ of all schemes, what Yoneda's Lemma says is the following:

- A scheme X determines the functor $h_X : \mathfrak{Sch} \to \mathbf{Ens}$ sending a scheme T to the set

$$h_X(T) = \hom_{\mathfrak{Sch}}(T, X).$$

- Conversely, a functor of the form h_X completely reconstructs X. This is because the functor which we denoted by F, acting by

  $$X \mapsto h_X,$$

  is "injective" (the correct word is *faithful*): roughly speaking, every object $h_X \in \mathrm{Ob}\,\underline{\mathrm{hom}}(\mathfrak{Sch}, \mathbf{Ens})$ comes from a unique scheme $X \in \mathfrak{Sch}$.

The functor h_X is usually called the *functor of points* of X. Indeed, a morphism $T \to X$ is, by definition, a T-valued point of the scheme X.

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